Generalized Linear Covariance Analysis

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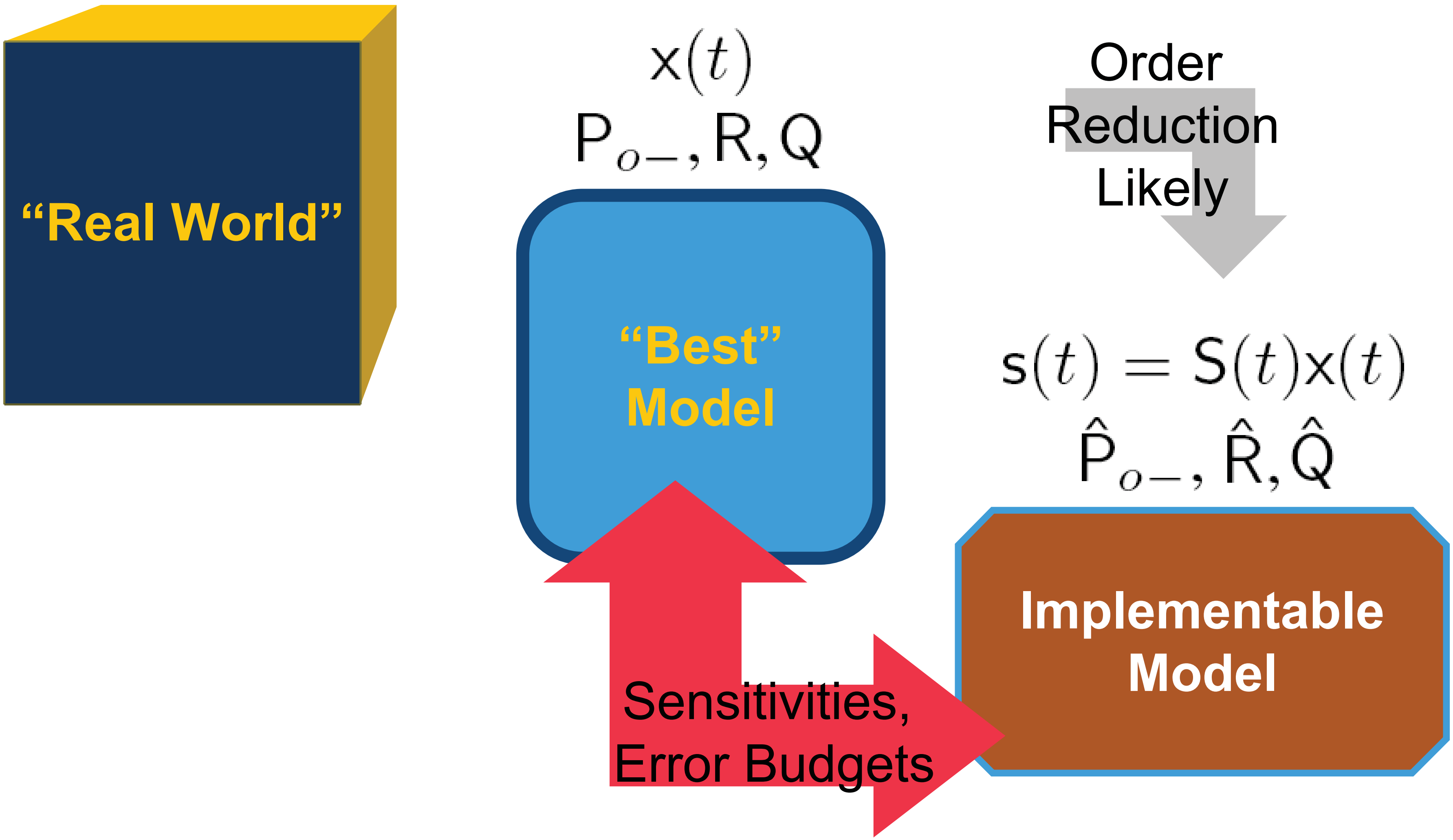
Motivation



Some Background

- Jazwinski (1970) - “solve-for/consider” state framework; assumes sequential filter
- Gelb (1974) - “true/filter” state framework (“true state” is “difference” between filter state and truth state); assumes sequential filter
- Maybeck (1979) - variation on Gelb; uses true (linear) state, rather than deviation from filter state
- Markley, et al. (1988) - uses solve/for consider framework, explicitly models contributions from a priori, measurement noise, and process noise for batch and sequential
- Tapley, et al. (2004) - uses solve-for/consider framework for batch and sequential

Present Work

- Update to Markley et al. ('88,'89)
- Explicitly addresses sensitivity of solve-fors with respect to considers in context of '88 paper's covariance partitions
- Explicitly addresses "postdiction" in the batch framework
- Applies method to integrated orbit/attitude problem
- Includes some new ways to examine output

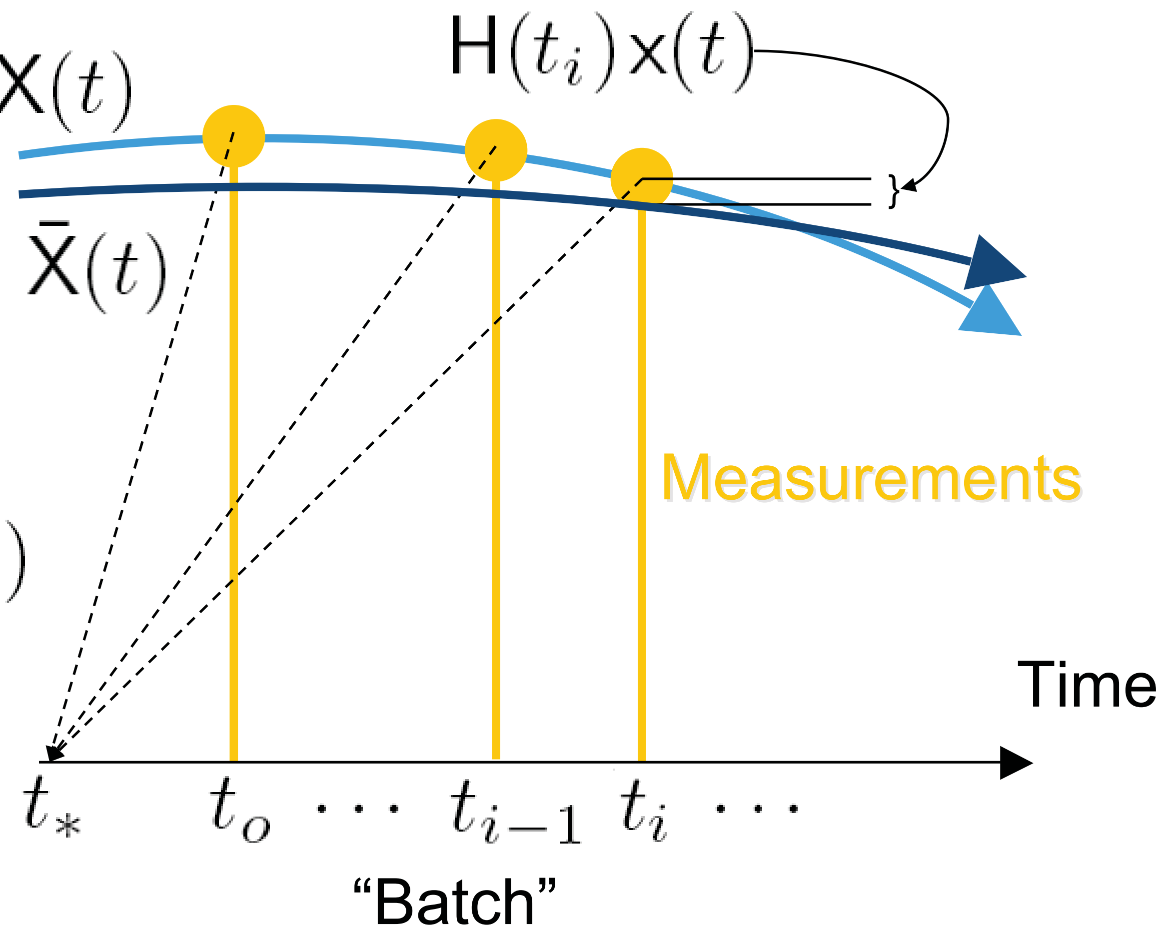
Context …

Actual Trajectory: $\mathsf{X}(t)$

Reference Trajectory: $\bar{\mathsf{X}}(t)$

<u>Linear Deviations:</u>

$$\mathsf{x}(t) = \mathsf{X}(t) - \bar{\mathsf{X}}(t)$$



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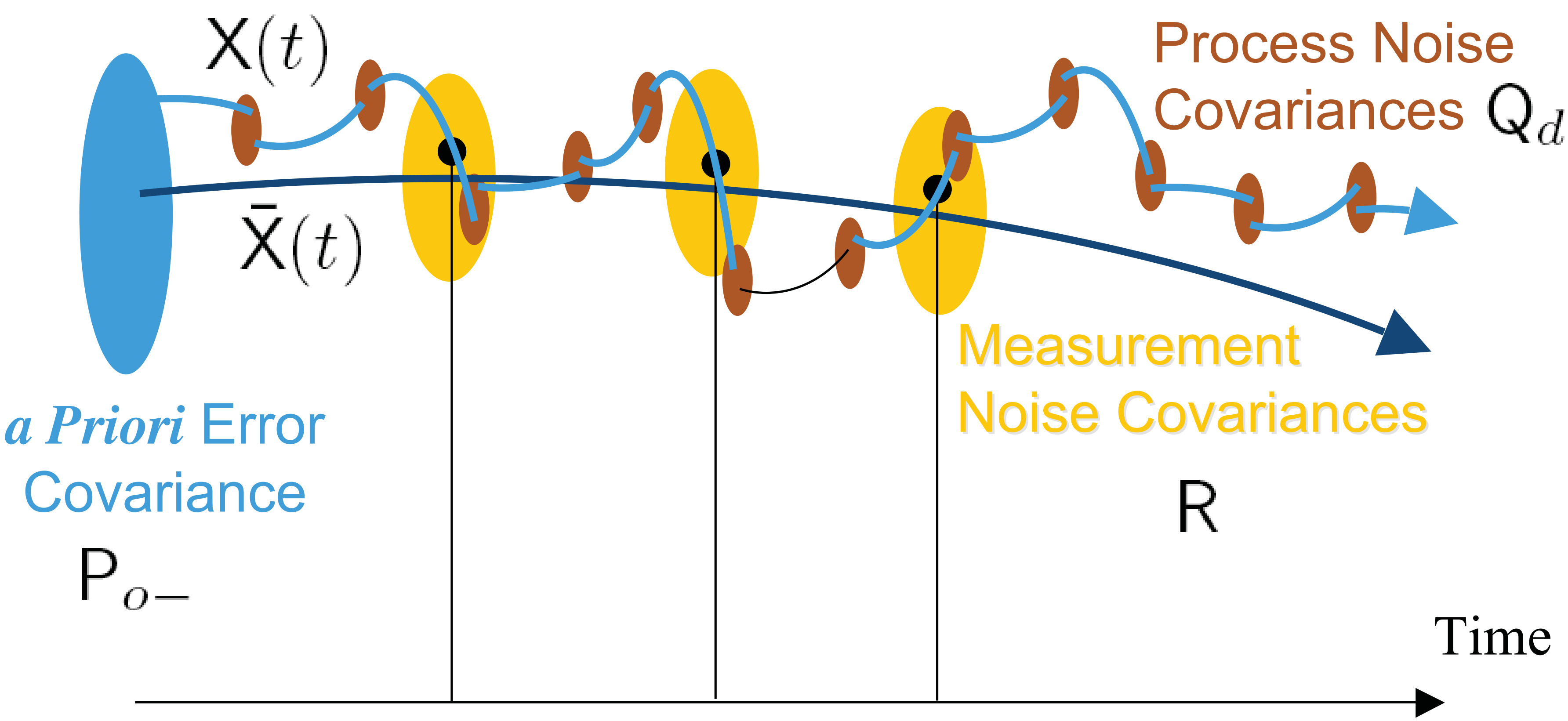
Context, Continued



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Solve-for and Consider State Mapping

Solve-For States

"Consider" States

$$\mathsf{s}(t) = \mathsf{S}(t)\mathsf{x}(t), \quad \mathsf{c}(t) = \mathsf{C}(t)\mathsf{x}(t)$$

Inverse Mapping:

$$\mathsf{M}(t) = \begin{bmatrix} \mathsf{S}(t) \\ \mathsf{C}(t) \end{bmatrix}, \quad \mathsf{M}^{-1}(t) = [\tilde{\mathsf{S}}(t),\ \tilde{\mathsf{C}}(t)]$$

$$\mathsf{x}(t) = \tilde{\mathsf{S}}(t)\mathsf{s}(t) + \tilde{\mathsf{C}}(t)\mathsf{c}(t)$$

Covariance Partitions

$$\mathsf{P} = \mathsf{P}_a + \mathsf{P}_v + \mathsf{P}_w, \quad \hat{\mathsf{P}} = \hat{\mathsf{P}}_a + \hat{\mathsf{P}}_v + \hat{\mathsf{P}}_w$$

Effect of *a Priori* Errors: $\Delta \mathsf{P}_a = \mathsf{S}\mathsf{P}_a\mathsf{S}^\mathsf{T} - \hat{\mathsf{P}}_a$

Effect of Measurement Noise: $\Delta \mathsf{P}_v = \mathsf{S}\mathsf{P}_v\mathsf{S}^\mathsf{T} - \hat{\mathsf{P}}_v$

Effect of Process Noise: $\Delta \mathsf{P}_w = \mathsf{S}\mathsf{P}_w\mathsf{S}^\mathsf{T} - \mathsf{P}_w$

N.B.: Δ's are not necessarily positive or negative

Auxiliary Notation

Solve-For Mapping of Dynamics and Measurements Partials

$$\mathsf{\Phi}_{ss}(t_i, t_o) = \mathsf{S}(t_i)\mathsf{\Phi}(t_i, t_o)\tilde{\mathsf{S}}(t_o), \quad \mathsf{H}_s(t_i) = \mathsf{H}(t_i)\tilde{\mathsf{S}}(t_i)$$

Process Noise Covariance Matrices

$$\hat{\mathsf{Q}}_d(t_i, t_{i-1}) = \int_{t_{i-1}}^{t_i} \mathsf{\Phi}_{ss}(t_i, \tau)\hat{\mathsf{Q}}(\tau)\mathsf{\Phi}_{ss}^{\mathsf{T}}(t_i, \tau)\mathrm{d}\tau$$

$$\mathsf{Q}_d(t_i, t_{i-1}) = \int_{t_{i-1}}^{t_i} \mathsf{\Phi}(t_i, \tau)\mathsf{Q}(\tau)\mathsf{\Phi}^{\mathsf{T}}(t_i, \tau)\mathrm{d}\tau$$

Sequential Filter Propagation

Only the "*a* Partition" receives the *a Priori* Errors …

$$\hat{P}_a(t_i^-) = \Phi_{ss}(t_i, t_{i-1})\hat{P}_a(t_{i-1}^+)\Phi_{ss}^{\mathsf{T}}(t_i, t_{i-1}), \qquad \hat{P}_a(t_1^-) = \hat{P}_{o-}$$

$$P_a(t_i^-) = \Phi(t_i, t_{i-1})P_a(t_{i-1}^+)\Phi^{\mathsf{T}}(t_i, t_{i-1}), \qquad P_a(t_1^-) = P_{o-}$$

$$\hat{P}_v(t_i^-) = \Phi_{ss}(t_i, t_{i-1})\hat{P}_v(t_{i-1}^+)\Phi_{ss}^{\mathsf{T}}(t_i, t_{i-1}), \qquad \hat{P}_v(t_1^-) = 0$$

$$P_v(t_i^-) = \Phi(t_i, t_{i-1})P_v(t_{i-1}^+)\Phi^{\mathsf{T}}(t_i, t_{i-1}), \qquad P_v(t_1^-) = 0$$

$$\hat{P}_w(t_i^-) = \Phi_{ss}(t_i, t_{i-1})\hat{P}_w(t_{i-1}^+)\Phi_{ss}^{\mathsf{T}}(t_i, t_{i-1}) + \hat{Q}_d(t_i, t_{i-1}), \qquad \hat{P}_w(t_1^-) = 0$$

$$P_w(t_i^-) = \Phi(t_i, t_{i-1})P_w(t_{i-1}^+)\Phi^{\mathsf{T}}(t_i, t_{i-1}) + Q_d(t_i, t_{i-1}), \qquad P_w(t_1^-) = 0$$

… and only the "*w* Partition" receives the Process Noise

Sequential Filter Update

The Gain is computed from the Formal Solve-for Covariance:

$$\mathsf{K}_i = \hat{\mathsf{P}}(t_i^-)\mathsf{H}_s^\mathsf{T}(t_i)[\mathsf{H}_s(t_i)\hat{\mathsf{P}}(t_i^-)\mathsf{H}_s^\mathsf{T}(t_i) + \hat{\mathsf{R}}(t_i)]^{-1}$$

$$\hat{\mathsf{P}}_a(t_i^+) = [\mathsf{I} - \mathsf{K}_i\mathsf{H}_s(t_i)]\hat{\mathsf{P}}_a(t_i^-)[\mathsf{I} - \mathsf{K}_i\mathsf{H}_s(t_i)]^\mathsf{T}$$

$$\mathsf{P}_a(t_i^+) = [\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]\mathsf{P}_a(t_i^-)[\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]^\mathsf{T}$$

$$\hat{\mathsf{P}}_v(t_i^+) = [\mathsf{I} - \mathsf{K}_i\mathsf{H}_s(t_i)]\hat{\mathsf{P}}_v(t_i^-)[\mathsf{I} - \mathsf{K}_i\mathsf{H}_s(t_i)]^\mathsf{T} + \mathsf{K}_i\boxed{\hat{\mathsf{R}}(t_i)}\mathsf{K}_i^\mathsf{T}$$

$$\mathsf{P}_v(t_i^+) = [\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]\mathsf{P}_v(t_i^-)[\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]^\mathsf{T} + \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\boxed{\mathsf{R}(t_i)}\mathsf{K}_i^\mathsf{T}\tilde{\mathsf{S}}^\mathsf{T}(t_i)$$

$$\hat{\mathsf{P}}_w(t_i^+) = [\mathsf{I} - \mathsf{K}_i\mathsf{H}_s(t_i)]\hat{\mathsf{P}}_w(t_i^-)[\mathsf{I} - \mathsf{K}_i\mathsf{H}_s(t_i)]^\mathsf{T}$$

$$\mathsf{P}_w(t_i^+) = [\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]\mathsf{P}_w(t_i^-)[\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]^\mathsf{T}$$

Only the "v Partition" receives the Measurement Noise

Batch Estimator without Process Noise

We write the batch update in a form resembling that of the sequential filter to isolate the contributions of the *a priori* error and the measurement noise:

$$\mathsf{K}_i = \left[\hat{\mathsf{P}}_{*-}^{-1} + \textstyle\sum_j \Phi_{ss}^{\mathsf{T}}(t_j, t_*)\mathsf{H}_s^{\mathsf{T}}(t_j)\hat{\mathsf{R}}_j^{-1}\mathsf{H}_s(t_j)\Phi_{ss}(t_j, t_*)\right]^{-1}\Phi_{ss}^{\mathsf{T}}(t_i, t_*)\mathsf{H}_s^{\mathsf{T}}(t_i)\hat{\mathsf{R}}_i^{-1}$$

$$\hat{\mathsf{P}}_a(t_*^+) = \left[\mathsf{I} - \textstyle\sum_i \mathsf{K}_i\mathsf{H}_s(t_i)\Phi_{ss}(t_i, t_*)\right]\boxed{\hat{\mathsf{P}}_{*-}}\left[\mathsf{I} - \textstyle\sum_i \mathsf{K}_i\mathsf{H}_s(t_i)\Phi_{ss}(t_i, t_*)\right]^{\mathsf{T}}$$

$$\mathsf{P}_a(t_*^+) = \left[\mathsf{I} - \textstyle\sum_i \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)\Phi(t_i, t_*)\right]\boxed{\mathsf{P}_{*-}}\left[\mathsf{I} - \textstyle\sum_i \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)\Phi(t_i, t_*)\right]^{\mathsf{T}}$$

$$\hat{\mathsf{P}}_v(t_*^+) = \textstyle\sum_i \mathsf{K}_i\boxed{\hat{\mathsf{R}}(t_i)}\mathsf{K}_i^{\mathsf{T}}, \quad \mathsf{P}_v(t_*^+) = \textstyle\sum_i \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\boxed{\mathsf{R}(t_i)}\mathsf{K}_i^{\mathsf{T}}\tilde{\mathsf{S}}^{\mathsf{T}}(t_i)$$

We associate the *a priori* error with an anchor time that is not required to precede the measurement batch.

Effect of Process Noise on Batch Estimator

The batch estimator does not model process noise, but if it is present, it will induce an error:

$$\mathsf{P}_w(t_*^+) = \Upsilon \begin{bmatrix} \mathsf{Q}_d(t_*;t_1,t_1) & \mathsf{Q}_d(t_*;t_1,t_2) & \cdots \\ \mathsf{Q}_d(t_*;t_2,t_1) & \mathsf{Q}_d(t_*;t_2,t_2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \Upsilon^{\mathsf{T}}$$

$$\Upsilon = \begin{bmatrix} \tilde{\mathsf{S}}(t_1)\mathsf{K}_1\mathsf{H}(t_1)\Phi(t_1,t_*) & \tilde{\mathsf{S}}(t_2)\mathsf{K}_2\mathsf{H}(t_2)\Phi(t_2,t_*) & \cdots \end{bmatrix}$$

Generalized Process Noise Covariance

Process noise enters the innovations, and thus must be mapped to the anchor time:

$$\mathsf{Q}_d(t_*; t_i, t_j) = \begin{cases} \int_{t_*}^{\min(t_i,t_j)} \Phi(t_i,\tau)\mathsf{Q}(\tau)\Phi^\mathsf{T}(t_j,\tau)\mathrm{d}\tau & t_* < t_i, t_* < t_j, \\ \int_{\max(t_i,t_j)}^{t_*} \Phi(t_i,\tau)\mathsf{Q}(\tau)\Phi^\mathsf{T}(t_j,\tau)\mathrm{d}\tau & t_i < t_*, t_j < t_*, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathsf{Q}_d(t_i, t_*)\Phi^\mathsf{T}(t_j, t_i) & t_* < t_i \le t_j, \\ \Phi(t_i, t_j)\mathsf{Q}_d(t_j, t_*) & t_* \le t_j \le t_i, \\ \Phi(t_i, t_j)\mathsf{Q}_d(t_*, t_j) & t_i \le t_j < t_*, \\ \mathsf{Q}_d(t_*, t_i)\Phi^\mathsf{T}(t_j, t_i) & t_j \le t_i < t_*, \\ 0 & \text{otherwise} \end{cases}$$

If the anchor time is between a pair of measurements, their process noise contributions do not overlap, so the expected contribution to the covariance is zero

Sensitivity to Individual *a Priori* Parameters

- **In this work, a "sensitivity" is a matrix of partial derivatives of some parameters of interest with respect to others (cf. Gelb, where "sensitivity" refers to a covariance matrix)**
- **The "ΔP's" give an indication of sensitivities of solve-for covariances to groups of errors (*a priori*, measurement noise, and process noise)**
- **Often we'd also like to know the sensitivity of each solve-for at any point in time to each individual *a priori* solve-for or consider error**

Sensitivity of solve-fors with respect to *a priori*'s (sequential form):

$$\Sigma_a(t_i) = [\mathsf{I} - \tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)]\Phi(t_i, t_{i-1})\Sigma_a(t_{i-1}), \quad \Sigma_a(t_o) = [\tilde{\mathsf{S}}(t_o), \tilde{\mathsf{C}}(t_o)]$$

Sensitivity of solve-fors with respect to *a priori*'s (batch form):

$$\Sigma_a(t) = \mathsf{S}(t)\Phi(t, t_*)[\mathsf{I} - \textstyle\sum_{i=1}^{k}\tilde{\mathsf{S}}(t_i)\mathsf{K}_i\mathsf{H}(t_i)\Phi(t_i, t_*)][\tilde{\mathsf{S}}(t_*), \tilde{\mathsf{C}}(t_*)]$$

Example: Entry Problem

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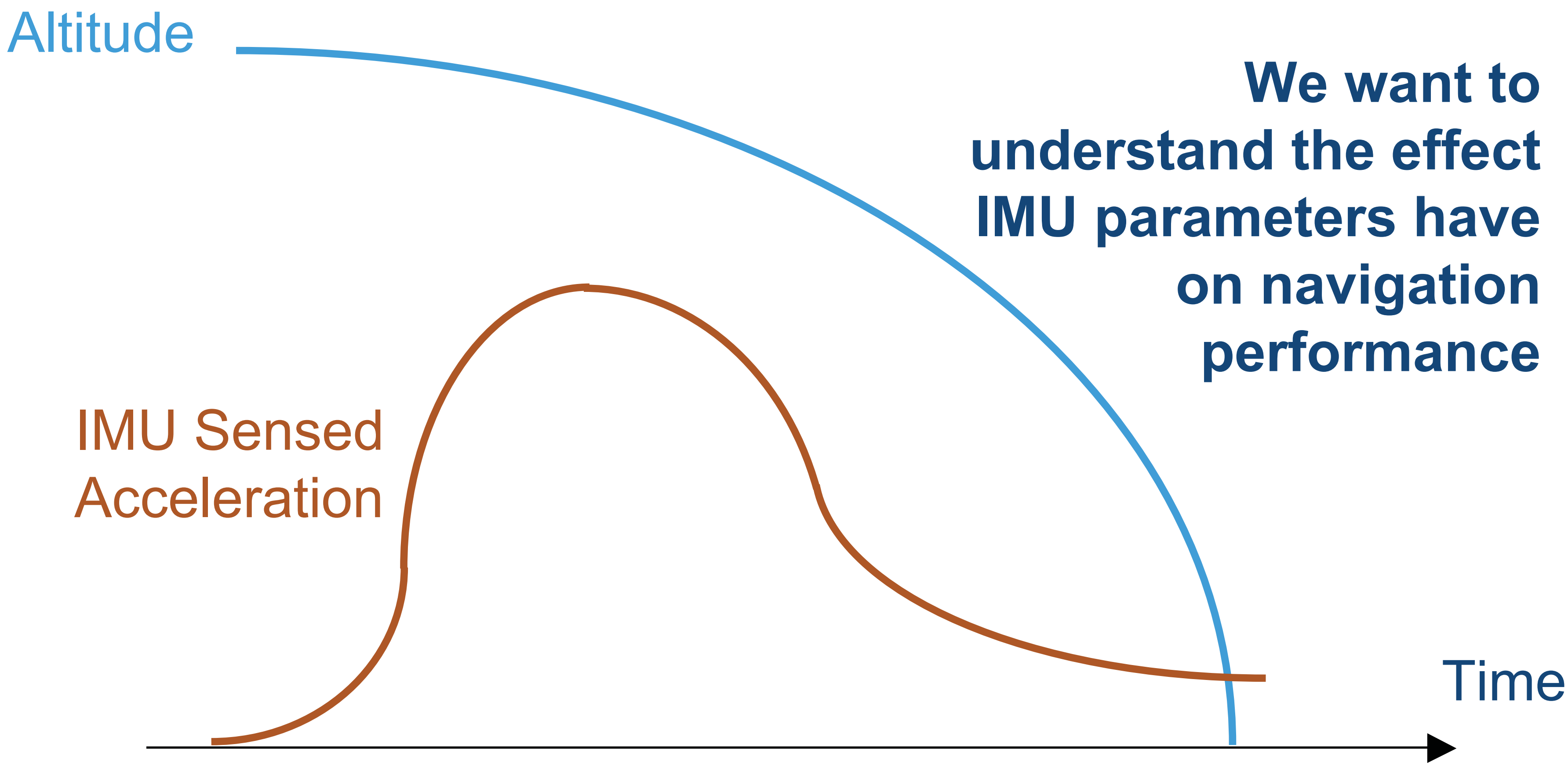

We want to understand the effect IMU parameters have on navigation performance

IMU Model

$$\mathsf{x}_g = [\boldsymbol{\theta}_{\mathcal{C}}^{\mathsf{T}}, \mathbf{b}_{g\mathcal{C}}^{\mathsf{T}}, \mathsf{s}_g^{\mathsf{T}}, \gamma_g^{\mathsf{T}}]^{\mathsf{T}}$$

<u>Gyro state</u>: angular misalignment, rate bias, scale factor, non-orthogonality

$$\begin{pmatrix} \frac{^{\mathcal{C}}d}{dt}\boldsymbol{\theta}_{\mathcal{C}} \\ \frac{^{\mathcal{C}}d}{dt}\mathbf{b}_{g\mathcal{C}} \\ \frac{d}{dt}\mathsf{s}_g \\ \frac{d}{dt}\gamma_g \end{pmatrix} = \begin{bmatrix} \mathsf{O}_{3\times 3} & \mathsf{I}_3 & \mathsf{D}(^{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{C}}^{\mathcal{C}}) & \mathsf{F}(^{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{C}}^{\mathcal{C}}) \\ & \mathsf{O}_{9\times 12} & & \end{bmatrix} \mathsf{x}_g + \begin{bmatrix} \mathsf{I}_3 \\ \mathsf{O}_{9\times 3} \end{bmatrix} \boldsymbol{\varepsilon}_\omega$$

$$\dot{\mathsf{x}}_g = \mathsf{A}_g \mathsf{x}_g + \mathsf{B}_g \boldsymbol{\varepsilon}_\omega,$$

$$\mathsf{x}_a = [\mathbf{b}_{a\mathcal{C}}^{\mathsf{T}}, \mathsf{s}_a^{\mathsf{T}}, \gamma_a^{\mathsf{T}}]^{\mathsf{T}}$$

<u>Accelerometer state</u>: acceleration bias, scale factor, non-orthogonality

$$\begin{aligned} \delta\mathbf{a}_{\mathcal{C}} &= \begin{bmatrix} \mathsf{I}_3 & \mathsf{D}(\mathbf{a}_{\mathcal{C}}) & \mathsf{F}(\mathbf{a}_{\mathcal{C}}) \end{bmatrix} \mathsf{x}_a \\ &= \mathsf{H}_a \mathsf{x}_a. \end{aligned}$$

Variance Sandpile: Y-Component of Inertial Position Error

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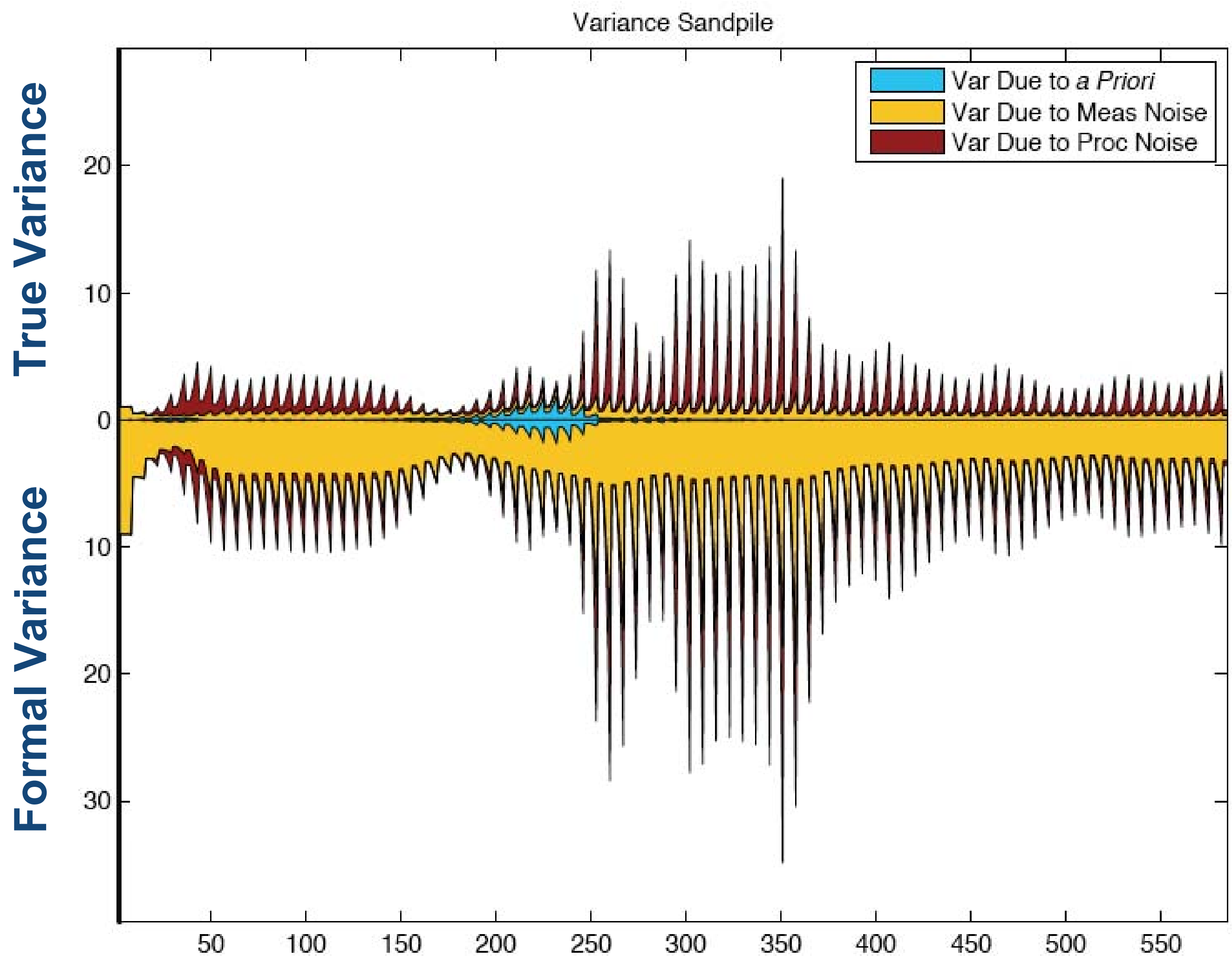

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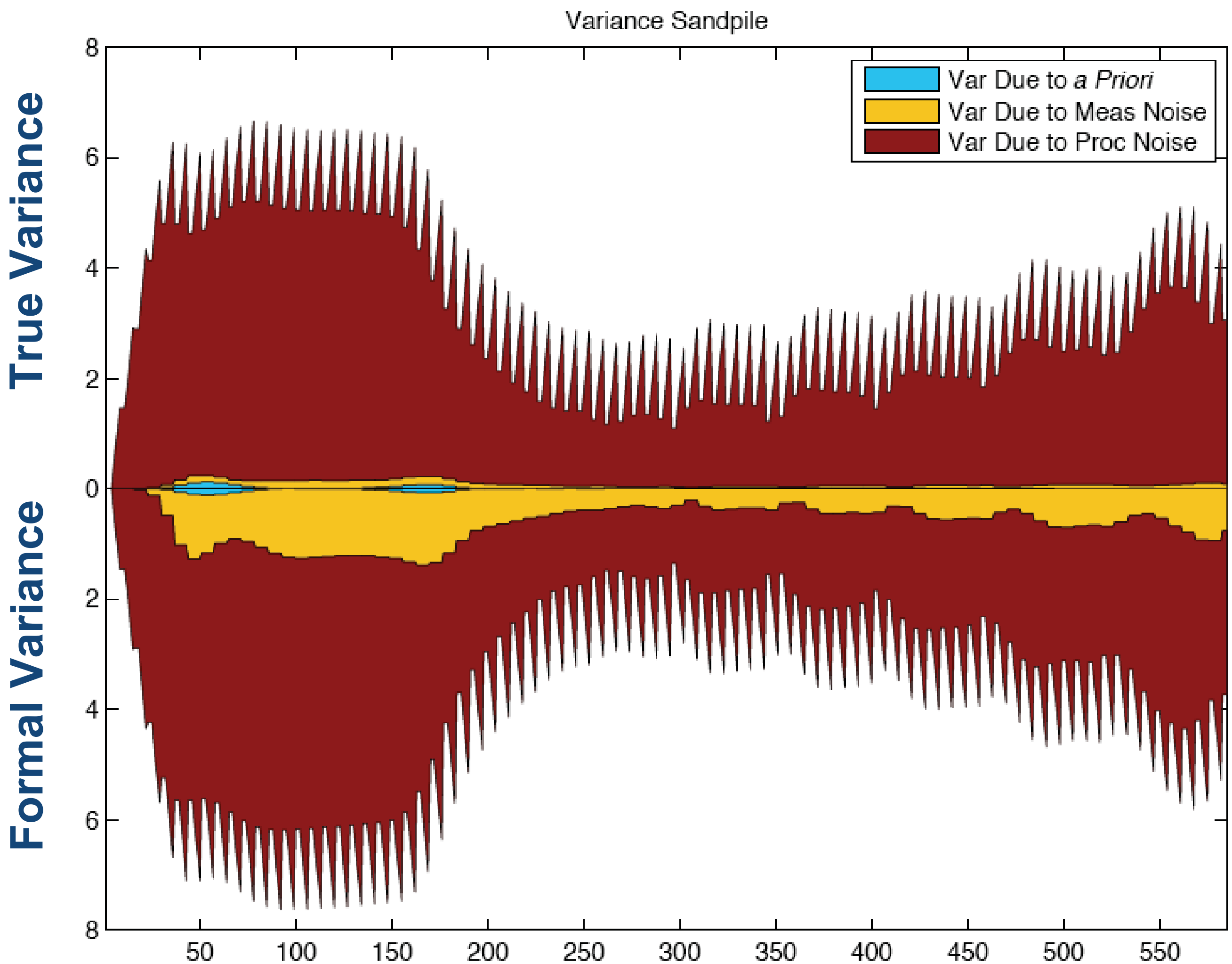
Variance Sandpile: Y-Component of Case-Fixed Gyro Angular Error



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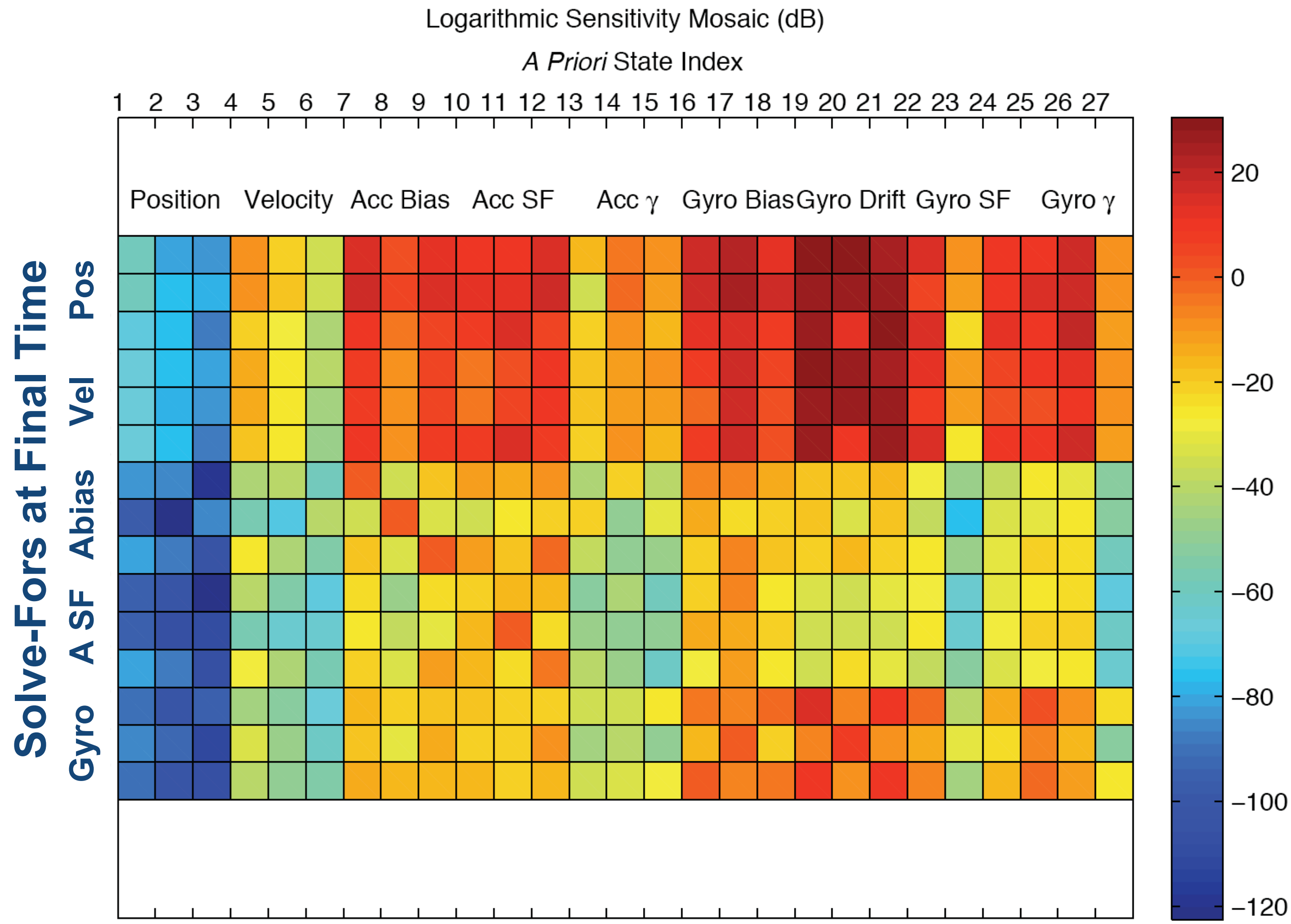
Sensitivity Mosaic



Summary

- Present work updates to Markley et al.
- Augments approach to sensitivity analysis
- Addresses "postdiction" in the batch framework
- Applies method to integrated orbit/attitude problem
- Includes some new ways to examine output

$$\Delta P_a = S P_a S^T - \hat{P}_a$$

$$\Delta P_v = S P_v S^T - \hat{P}_v$$

$$\Delta P_w = S P_w S^T - P_w$$

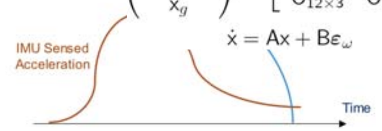
$$\mathsf{M}(t) = \begin{bmatrix} \mathsf{S}(t) \\ \mathsf{C}(t) \end{bmatrix}, \quad \mathsf{M}^{-1}(t) = [\tilde{\mathsf{S}}(t),\ \tilde{\mathsf{C}}(t)]$$

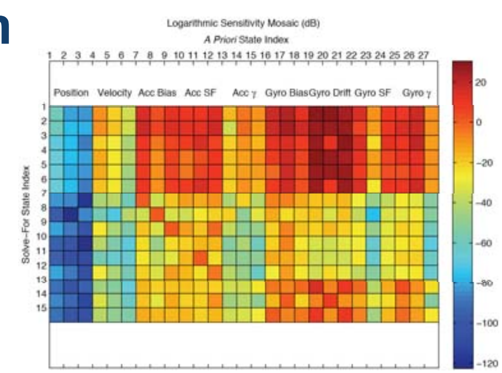
$$\mathsf{x}(t) = \tilde{\mathsf{S}}(t)\mathsf{s}(t) + \tilde{\mathsf{C}}(t)\mathsf{c}(t)$$

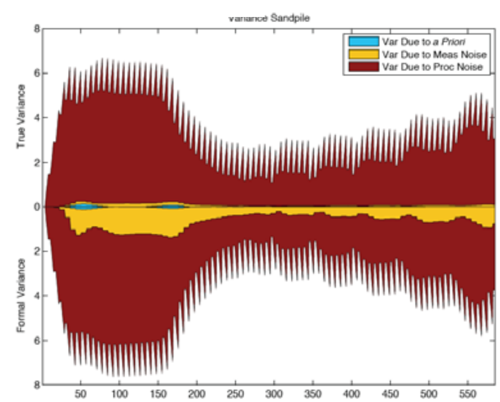
$$\mathsf{Q}_d(t_*; t_i, t_j) = \begin{cases} \int_{t_*}^{\min(t_i,t_j)} \Phi(t_i,\tau)\mathsf{Q}(\tau)\Phi^{\mathsf{T}}(t_j,\tau)\mathrm{d}\tau & t_* < t_i, t_* < t_j, \\ \int_{\max(t_i,t_j)}^{t_*} \Phi(t_i,\tau)\mathsf{Q}(\tau)\Phi^{\mathsf{T}}(t_j,\tau)\mathrm{d}\tau & t_i < t_*, t_j < t_*, \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} \frac{{}^{\mathcal{I}}d}{dt}\delta\mathbf{r}_{\mathcal{I}} \\ \frac{{}^{\mathcal{I}}d}{dt}\delta\mathbf{v}_{\mathcal{I}} \\ \dot{\mathsf{x}}_a \\ \dot{\mathsf{x}}_g \end{pmatrix} = \begin{bmatrix} \mathsf{O}_{3\times3} & \mathsf{I}_3 & \mathsf{O}_{3\times9} & \mathsf{O}_{3\times12} \\ \mathsf{G}(\mathbf{r}_{\mathcal{I}}) & \mathsf{O}_{3\times3} & \mathsf{M}_{\mathcal{C}}^{\mathcal{I}}\mathsf{H}_a & \mathsf{M}_{\mathcal{C}}^{\mathcal{I}}\mathsf{X}_{ag} \\ \mathsf{O}_{12\times3} & \mathsf{O}_{12\times3} & \mathsf{O}_{9\times9} & \mathsf{O}_{12\times12} \\ \mathsf{O}_{12\times3} & \mathsf{O}_{12\times3} & \mathsf{O}_{9\times9} & \mathsf{A}_g \end{bmatrix} \begin{pmatrix} \delta\mathbf{r}_{\mathcal{I}} \\ \delta\mathbf{v}_{\mathcal{I}} \\ \mathsf{x}_a \\ \mathsf{x}_g \end{pmatrix} + \begin{bmatrix} \mathsf{O}_{15\times3} \\ \mathsf{B}_g \end{bmatrix} \varepsilon_\omega$$

$$\dot{\mathsf{x}} = \mathsf{A}\mathsf{x} + \mathsf{B}\varepsilon_\omega$$







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INS Model

IMU sensed acceleration error integrates into postion and velocity errors according to the INS model:

$$\begin{pmatrix} \frac{{}^{\mathcal{I}}d}{dt}\delta\mathbf{r}_{\mathcal{I}} \\ \frac{{}^{\mathcal{I}}d}{dt}\delta\mathbf{v}_{\mathcal{I}} \\ \dot{\mathsf{x}}_a \\ \dot{\mathsf{x}}_g \end{pmatrix} = \begin{bmatrix} \mathsf{O}_{3\times3} & \mathsf{I}_3 & \mathsf{O}_{3\times9} & \mathsf{O}_{3\times12} \\ \mathsf{G}(\mathbf{r}_{\mathcal{I}}) & \mathsf{O}_{3\times3} & \mathsf{M}_{\mathcal{C}}^{\mathcal{I}}\mathsf{H}_a & \mathsf{M}_{\mathcal{C}}^{\mathcal{I}}\mathsf{X}_{ag} \\ \mathsf{O}_{12\times3} & \mathsf{O}_{12\times3} & \mathsf{O}_{9\times9} & \mathsf{O}_{12\times12} \\ \mathsf{O}_{12\times3} & \mathsf{O}_{12\times3} & \mathsf{O}_{9\times9} & \mathsf{A}_g \end{bmatrix} \begin{pmatrix} \delta\mathbf{r}_{\mathcal{I}} \\ \delta\mathbf{v}_{\mathcal{I}} \\ \mathsf{x}_a \\ \mathsf{x}_g \end{pmatrix} + \begin{bmatrix} \mathsf{O}_{15\times3} \\ \mathsf{B}_g \end{bmatrix} \varepsilon_\omega$$

$$\dot{\mathsf{x}} = \mathsf{A}\mathsf{x} + \mathsf{B}\varepsilon_\omega$$

where

$$\mathsf{X}_{ag} = [\mathbf{a}_{\mathcal{C}}^{\times}, \mathsf{O}_{3\times9}]$$

Example Problem Parameters

Simulation Parameter	Value	Units
Gravitational Constant	4.305×10^4	km^3/sec^2
Measurement Time Interval	2	sec
Estimation Parameter	**Standard Deviation**	**Units**
True Position Measurement Noise	0.305	m
Formal Position Measurement Noise	0.914	m
Initial Position Error	30.5	meters
Initial Velocity Error	3.05	cm/sec
Accelerometer Bias	60	μg
Accelerometer Scale Factor	500	ppm
Accelerometer Nonorthogonality	10	ppm
Initial Gyro Angular Error	42	arcsec
Gyro Bias Drift	0.01	deg/hr
Gyro Scale Factor	33	ppm
Gyro Nonorthogonality	20	ppm
Gyro Random Walk Intensity	0.025	$deg/hr^{1/2}$